Quantitative Trading as a Mathematical Science

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Abstract

- Quantitative trading is distinguishable from other trading methodologies like technical analysis and analysts' opinions because it uniquely provides justifications to trading strategies using mathematical reasoning. Put differently, quantitative trading is a science that trading strategies are proven statistically profitable or even optimal under certain assumptions. There are properties about strategies that we can deduce before betting the first $1, such as P&L distribution and risks. There are exact explanations to the success and failure of strategies, such as choice of parameters. There are ways to iteratively improve strategies based on experiences of live trading, such as making more realistic assumptions. These are all made possible only in quantitative trading because we have assumptions, models and rigorous mathematical analysis.
- Quantitative trading has proved itself to be a significant driver of mathematical innovations, especially in the areas of stochastic analysis and PDE-theory. For instances, we can compute the optimal timings to follow the market by solving a pair of coupled Hamilton–Jacobi–Bellman equations; we can construct sparse mean reverting baskets by solving semi-definite optimization problems with cardinality constraints and can optimally trade these baskets by solving stochastic control problems; we can identify statistical arbitrage opportunities by analyzing the volatility process of a stochastic asset at different frequencies; we can compute the optimal placements of market and limit orders by solving combined singular and impulse control problems which leads to novel and difficult to solve quasi-variational inequalities.

Speaker Profile

- Dr. Haksun Li
- CEO, NM LTD.
- (Ex-)Adjunct Professors, Industry Fellow, Advisor, Consultant with the National University of Singapore, Nanyang Technological University, Fudan University, the Hong Kong University of Science and Technology.
- Quantitative Trader/Analyst, BNPP, UBS
- Ph.D., Computer Science, University of Michigan Ann Arbor
- M.S., Financial Mathematics, University of Chicago
- B.S., Mathematics, University of Chicago

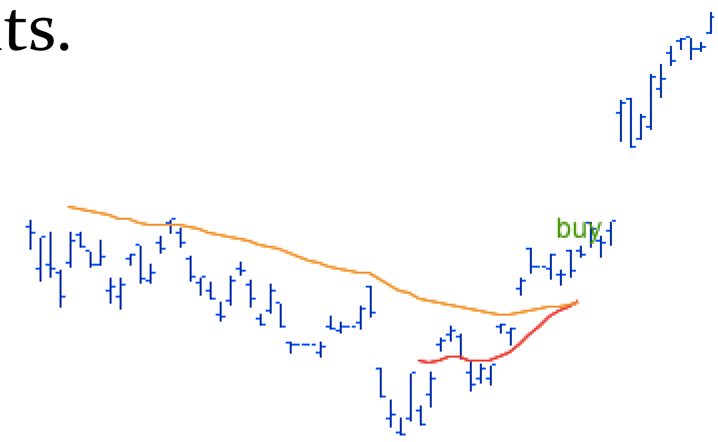
What is Quantitative Trading?

Quantitative Trading?

- Quantitative trading is the buying and selling of assets following the instructions computed from a set of **proven** mathematical models.
- The differentiation from other trading methodologies or the emphasis is on **how** a strategy is **proven** and not on what strategy is created.
- It applies (rigorous) mathematical reasoning in all steps during trading strategy construction from the start to the end.

Moving Average Crossover as a TA

- A popular TA signal: Moving Average Crossover.
  - A crossover occurs when a faster moving average (i.e. a shorter period moving average) crosses above/below a slower moving average (i.e. a longer period moving average); then you buy/sell.
- In most TA book, it is **never proven** only **illustrated** with an example of applying the strategy to a stock for a period of time to show profits.



Technical Analysis is Not Quantitative Trading

- TA books merely describes the mechanics of strategies but never prove them.
- Appealing to common sense is not a mathematical proof.
- Conditional probabilities of outcomes are seldom computed. (Lo, Mamaysky, & Wang, 2000)
- Application of TA is more of an art (is it?) than a science.
  - How do you choose the parameters?
- For any TA rule, you almost surely can find an asset and a period that the rule "works", given the large number of assets and many periods to choose from.

Fake Quantitative Models

- Data snooping
- Misuse of mathematics
- Assumptions cannot be quantified
- No model validation against the current regime
- Ad-hoc take profit and stop-loss
  - why 2?
- How do you know when a model is invalidated?
- Cannot explain winning and losing trades
- Cannot be analyzed (systematically)

The Quantitative Trading Research Process

NM Quantitative Trading Research Process

1. Translate a vague trading intuition (hypothesis) into a concrete mathematical model.
2. Translate the mathematical symbols and equations into a computer program.
3. Strategy evaluation.
4. Live execution for making money.

Step 1 - Modeling

- Where does a trading idea come from?
  - Ex-colleagues
  - Hearsays
  - Newspapers, books
  - TV, e.g., Moving Average Crossover (MA)
- A quantitative trading strategy is a math function, f, that at any given time, t, takes as inputs any information that the strategy cares and that is available, F_t, and gives as output the position to take, $f(t,F_t)$.

Step 2 - Coding

- The computer code enables analysis of the strategy.
  - Most study of a strategy cannot be done analytically.
  - We must resort to simulation.
- The same piece of code used for research and investigation should go straight into the production for live trading.
  - Eliminate the possibility of research-to-IT translation errors.

Step 3 – Evaluation/Justification

- Compute the properties of a trading strategy.
  - the P&L distribution
  - the holding time distribution
  - the stop-loss
  - the maximal drawdown
- http://redmine.numericalmethod.com/projects/public/repository/svn-algoquant/show/core/src/main/java/com/numericalmethod/algoquant/execution/performance

Step 4 - Trading

- Put in capitals incrementally.
- Install safety measures.
- Monitor the performance.
- Regime change detection.

Mathematical Analysis of Moving Average Crossover

Moving Average Crossover as a Quantitative Trading Strategy

- There are many mathematical justifications to Moving Average Crossover.
  - weighted Sum of lags of a time series
  - Kuo, 2002
- Whether a strategy is quantitative or not depends not on the strategy itself but
  - entirely on the process to construct it;
  - or, whether there is a scientific justification to prove it.

Step 1 - Modeling

- Two moving averages: slower (n) and faster (m).
- Monitor the crossovers.
- $B_t = \left(\frac{1}{m}\sum_{j=0}^{m-1} P_{t-j}\right) - \left(\frac{1}{n}\sum_{j=0}^{n-1} P_{t-j}\right), n > m$
- Long when $B_t \geq 0$.
- Short when $B_t < 0$.

How to Choose n and m?

- It is an art, not a science (so far).
- They should be related to the length of market cycles.
- Different assets have different n and m.
- Popular choices:
  - (250, 5)
  - (250 , 20)
  - (20 , 5)
  - (20 , 1)
  - (250 , 1)

Two Simplifications

- Reduce the two dimensional problem to a one dimensional problem.
  - Choose $m = 1$. We know that m should be small.
- Replace arithmetic averages with geometric averages.
  - This is so that we can work with log returns rather than prices.

GMA(n , 1)

- $B_t \geq 0$ iff $P_t \geq \left(\prod_{j=0}^{n-1} P_{t-j}\right)^{\frac{1}{n}}$
  - $R_t \geq -\sum_{j=1}^{n-2} \frac{n-(j+1)}{n-1} R_{t-j}$ (by taking log)
- $B_t < 0$ iff $P_t < \left(\prod_{j=0}^{n-1} P_{t-j}\right)^{\frac{1}{n}}$
  - $R_t < -\sum_{j=1}^{n-2} \frac{n-(j+1)}{n-1} R_{t-j}$ (by taking log)

What is n?

- $n = 2$
- $n = \infty$

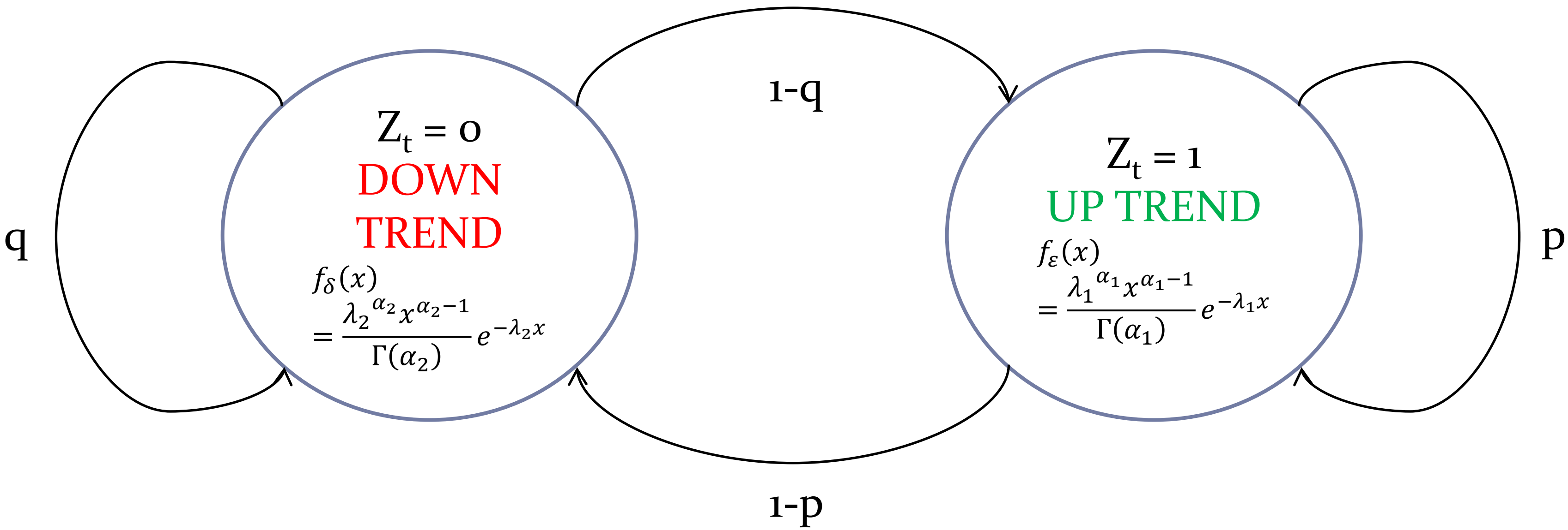
Acar Framework

- Acar (1993): to investigate the probability distribution of realized returns from a trading rule, we need
  - the explicit specification of the trading rule
  - the underlying stochastic process for asset returns
  - the particular return concept involved

Knight-Satchell-Tran Intuition

- Stock returns staying going up (down) depends on
  - the realizations of positive (negative) shocks
  - the persistence of these shocks
- Shocks are modeled by gamma processes.
  - Asymmetry
  - Fat tails
- Persistence is modeled by a Markov switching process.

Knight-Satchell-Tran Z_t



Knight-Satchell-Tran Process

- $R_t = \mu_l + Z_t \varepsilon_t - (1 - Z_t)\delta_t$
  - μ_l: long term mean of returns, e.g., 0
  - ε_t, δ_t: positive and negative shocks, non-negative, i.i.d
- $f_\varepsilon(x) = \frac{{\lambda_1}^{\alpha_1} x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} e^{-\lambda_1 x}$
- $f_\delta(x) = \frac{{\lambda_2}^{\alpha_2} x^{\alpha_2 - 1}}{\Gamma(\alpha_2)} e^{-\lambda_2 x}$

Step 3 – Evaluation/Justification

- Assume the long term mean is 0, $\mu_l = 0$.
- When $n = 2$,
  - $(B_t \geq 0) \equiv (R_t \geq 0) \equiv (Z_t = 1)$
  - $(B_t < 0) \equiv (R_t < 0) \equiv (Z_t = 0)$

GMA(2, 1) – Naïve MA Trading Rule

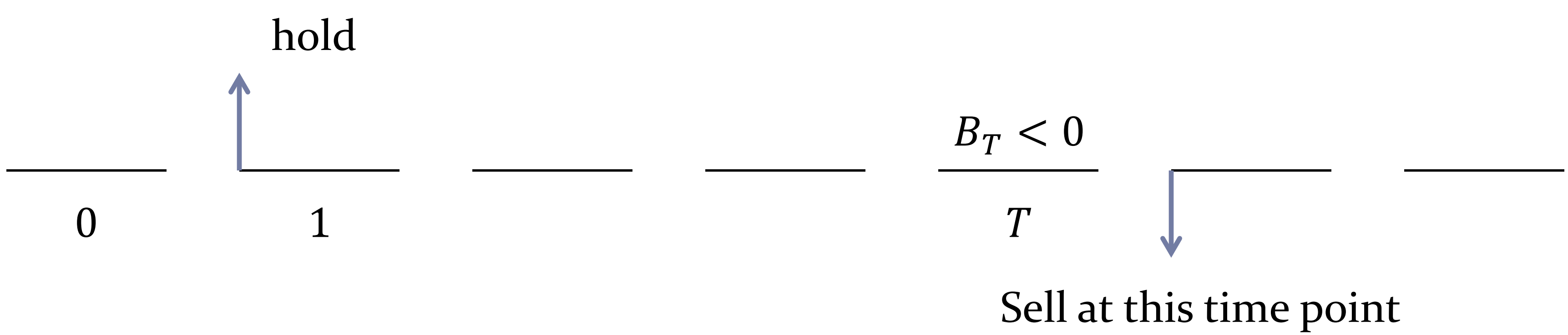
- Buy when the asset return in the present period is positive.
- Sell when the asset return in the present period is negative.

Naïve MA Conditions

- The expected value of the positive shocks to asset return >> the expected value of negative shocks.
- The positive shocks persistency >> that of negative shocks.

T Period Returns

- $RR_T = \sum_{t=1}^{T} R_t \times I_{\{B_{t-1} \geq 0\}}$



Holding Time Distribution

- $P(N = T)$
- $= P(B_T < 0, B_{T-1} \geq 0, \dots, B_1 \geq 0, B_0 \geq 0)$
- $= P(Z_T = 0, Z_{T-1} = 1, \dots, Z_1 = 1, Z_0 = 1)$
- $= P(Z_T = 0, Z_{T-1} = 1, \dots, Z_1 = 1 | Z_0 = 1) P(Z_0 = 1)$
- $= \begin{cases} \Pi p^{T-1}(1-p), \ \mathrm{T} \geq 1 \\ 1 - \Pi, \ \mathrm{T}=0 \end{cases}$

- Stationary state probability:
  - $\Pi = \frac{1-q}{2-p-q}$

Conditional Returns Distribution (1)

- $\Phi_{RR_T|N=T}(s) = \mathrm{E}\left[e^{\left\{i\left[\sum_{t=1}^{T} R_t \times I_{\{B_{t-1}\geq 0\}}\right]s\right\}} | N = T\right]$
- $= \mathrm{E}\left[e^{\left\{i\left[\sum_{t=1}^{T} R_t \times I_{\{B_{t-1}\geq 0\}}\right]s\right\}} | B_T < 0, B_{T-1} \geq 0, \dots, B_0 \geq 0\right]$
- $= \mathrm{E}\left[e^{\left\{i\left[\sum_{t=1}^{T} R_t\right]s\right\}} | Z_T = 0, Z_{T-1} = 1, \dots, Z_1 = 1\right]$
- $= \mathrm{E}\left[e^{\{i[\varepsilon_1 + \cdots + \varepsilon_{T-1} - \delta_T]s\}}\right]$
- $= \begin{cases} {\Phi_\varepsilon}^{T-1}(s)\Phi_\delta(-s), & \mathrm{T} \geq 1 \\ \Phi_\delta(-s), & \mathrm{T} = 0 \end{cases}$

Unconditional Returns Distribution (2)

- $\Phi_{RR_T}(s) = \sum_{T=0}^{\infty} \mathrm{E}\left[e^{\left\{i\left[\sum_{t=1}^{T} R_t \times I_{\{B_{t-1} \geq 0\}}\right]s\right\}} | N = T\right] P(N = T)$
- $= \sum_{T=1}^{\infty} \Pi p^{T-1}(1-p){\Phi_{\varepsilon}}^{T-1}(s)\Phi_{\delta}(-s) + (1-\Pi)\Phi_{\delta}(-s)$
- $= (1-\Pi)\Phi_{\delta}(-s) + \Pi(1-p)\frac{\Phi_{\delta}(-s)}{1-p\Phi_{\varepsilon}(s)}$

Expected Returns

- $\mathrm{E}(RR_T) = -i\Phi_{RR_T}{}'(0)$
- $= \frac{1}{1-p}\{\Pi p\mu_\varepsilon - (1-p)\mu_\delta\}$
- When is the expected return positive?
  - $\mu_\varepsilon \geq \frac{1-p}{\Pi p}\mu_\delta$, shock impact
  - $\mu_\varepsilon \gg \mu_\delta$, shock impact
  - $\Pi p \geq 1 - p$, if $\mu_\varepsilon \approx \mu_\delta$, persistence

GMA(∞,1) Rule

- $P_t \geq \left(\prod_{j=0}^{n-1} P_{t-j}\right)^{\frac{1}{n}}$
- $\ln P_t \geq \frac{1}{n}\sum_{j=0}^{n-1} \ln P_{t-j}$
- $\ln P_t \geq \mu_1$

GMA(∞,1) Expected Returns

- $\Phi_{RR_T}(s) =$
  $(1-\Pi)q[\Phi_\delta(s) + \Phi_\delta(-s)] +$
  $[1-p(1-\Pi)][\Phi_\varepsilon(s) + \Phi_\varepsilon(-s)]$
- $\mathrm{E}(RR_T) = -[1-p(1-\Pi)][\mu_\varepsilon + \mu_\delta]$

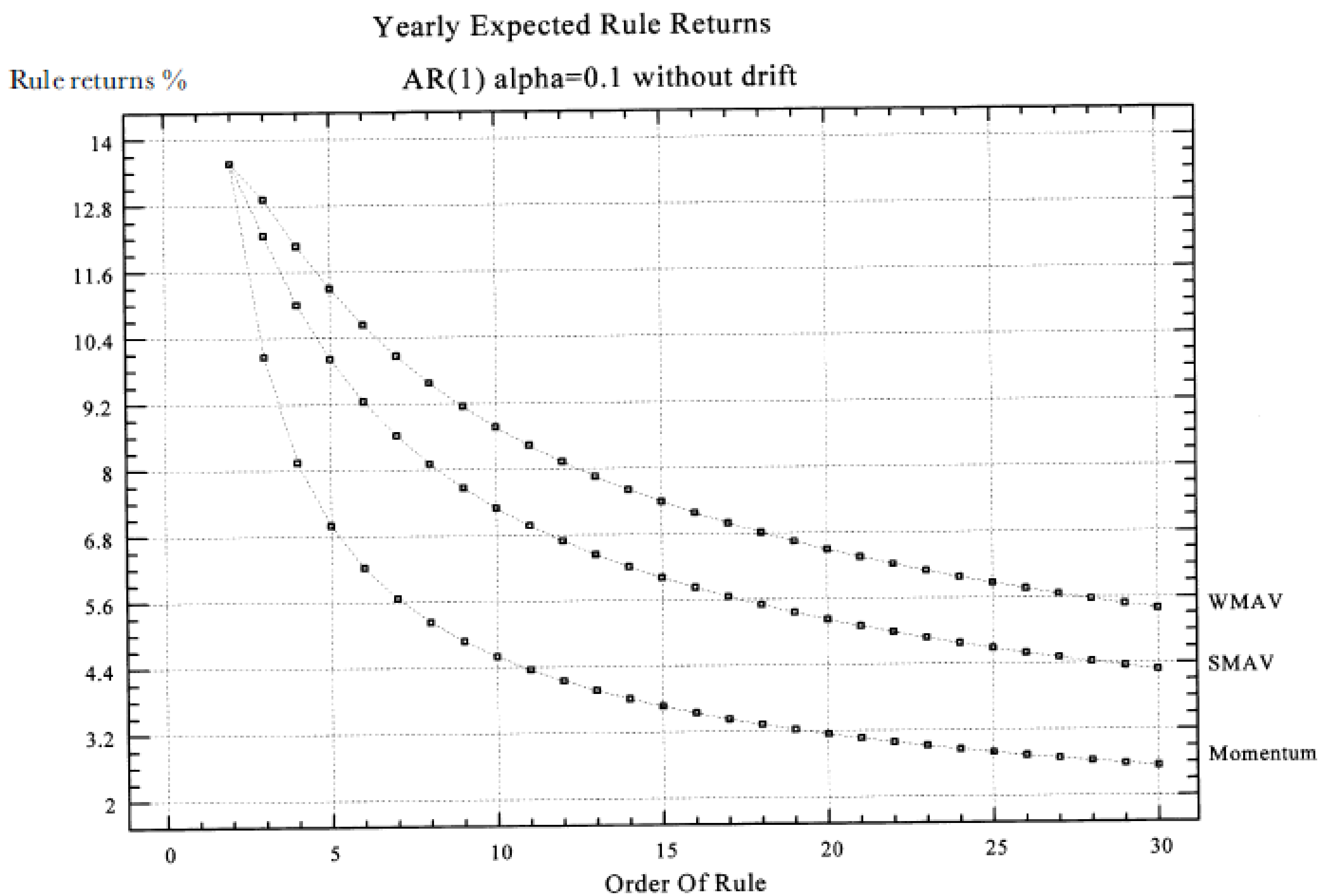
MA Using the Whole History

- An investor will always expect to lose money using GMA(∞,1)!
- An investor loses the least amount of money when the return process is a random walk.

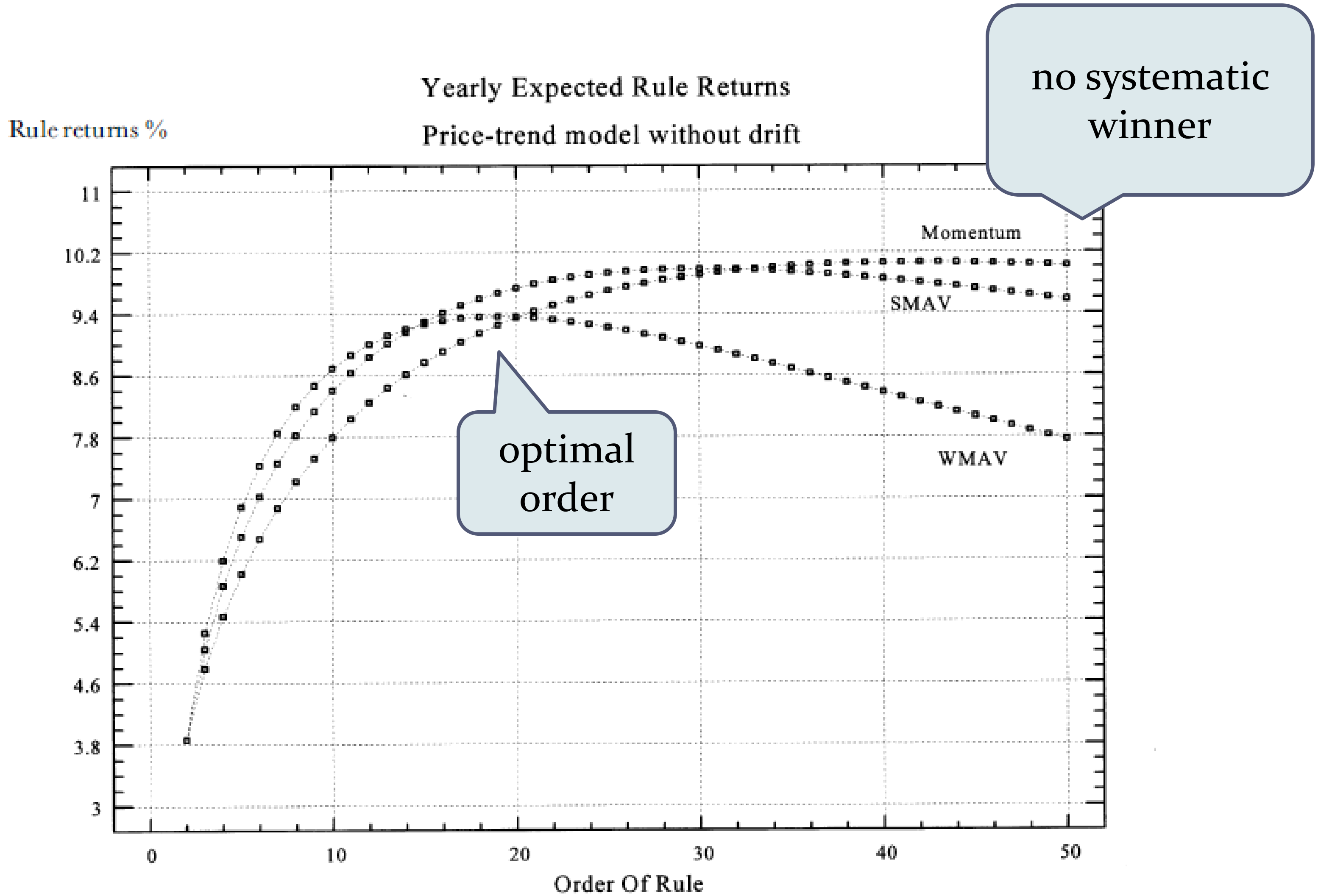
Optimal MA Parameters

- So, what are the optimal n and m?

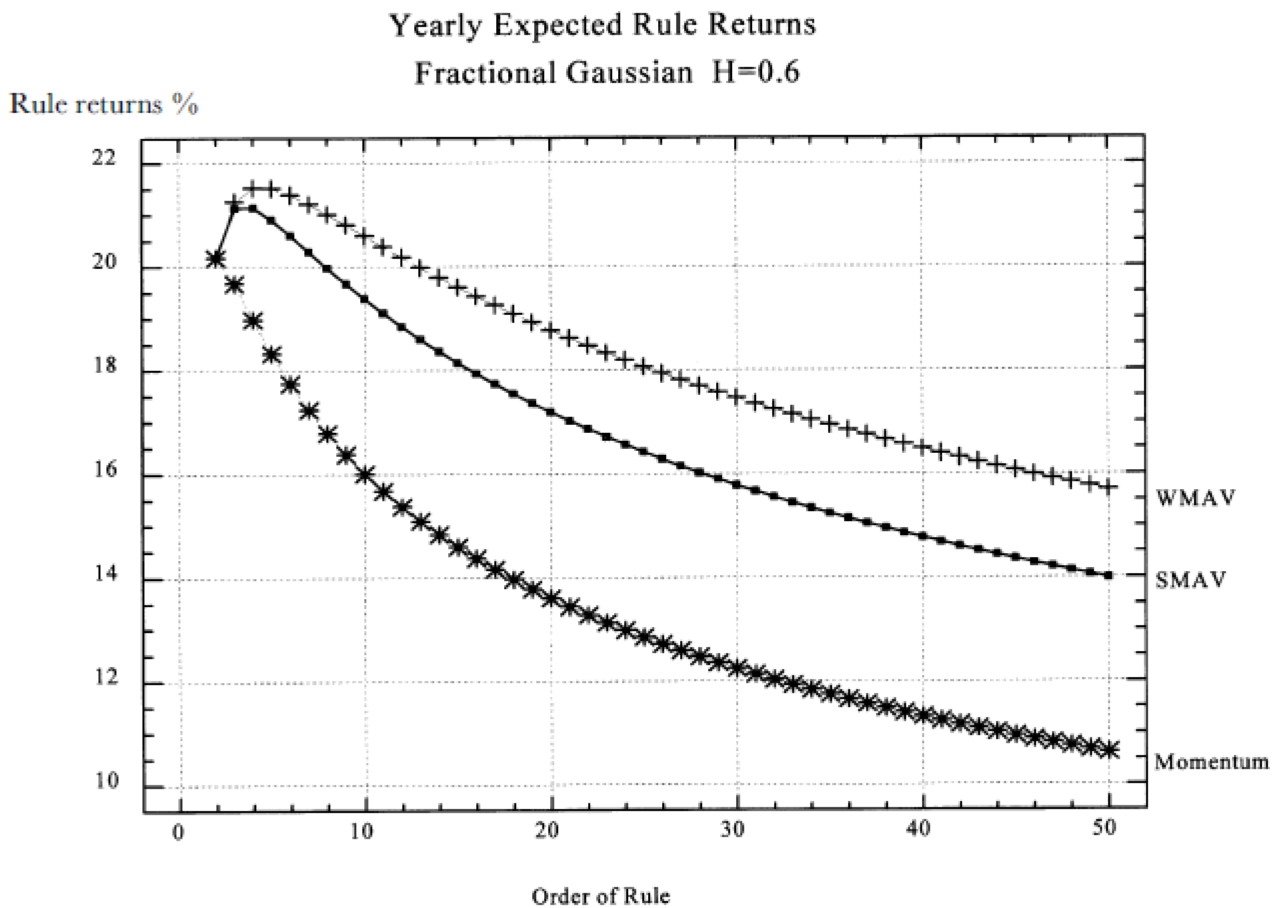
Step 2: AR(1)


Yearly Expected Rule Returns
AR(1) alpha=0.1 without drift
Rule returns %
14
12.8
11.6
10.4
9.2
8
6.8
5.6
4.4
3.2
2
0
5
10
15
20
25
30
Order Of Rule
WMAV
SMAV
Momentum

Step 2 : ARMA(1, 1)



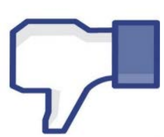
Step 2 : ARIMA(0, d, 0)



Live Results of Quantitative Trading Strategies

Unique Guiding Principle

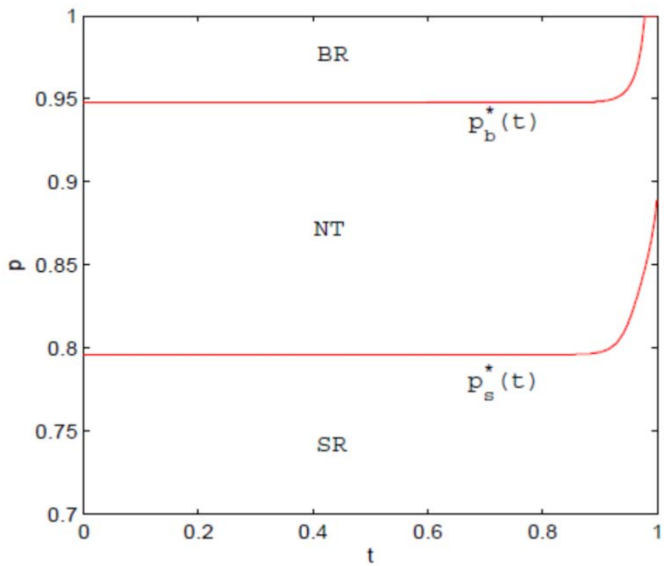
- What Others Do:
  - Start with a trading strategy.
  - Find the data that the strategy works.
- Result:
  - Paper P&L looks good.
  - Live P&L depends on luck.
- Trading strategies are results of a non-scientific, a pure data snooping process.

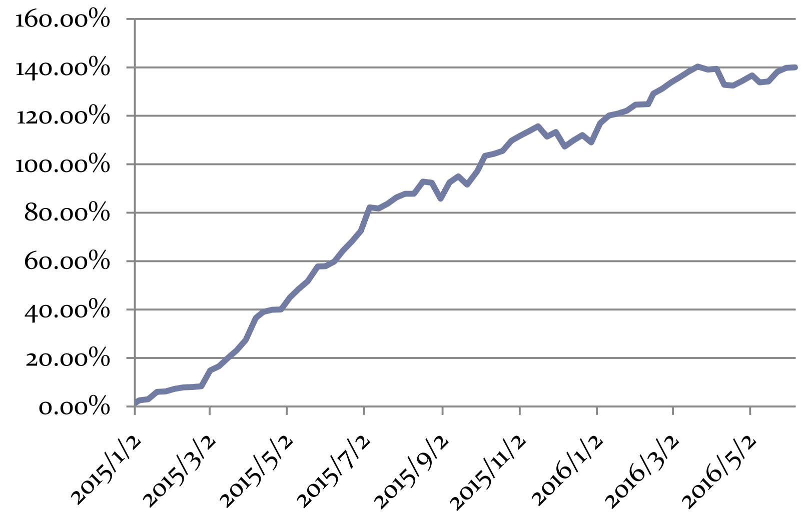


- What We Do:
  - Start with simple assumptions about the market.
  - ***Compute*** the optimal trading strategy given the assumptions.
- Result:
  - Can mathematically prove that no other strategy will work better in the same market conditions.
- Trading strategies are results of a scientific process.

Optimal Trend Following (TREND)

- We make assumptions that the market is a two (or three) state model. The market state is either up, down, (or sideway).
- In each state, we assume a random walk with positive, negative, or zero drift.
- We use math to compute what the best thing to do is in each of the states.
- We estimate the conditional probability, p, of that the market is going up given all the available information.
- When p is big enough, i.e., most certainly that the market is going up, we buy.





Result:

trading period	2015/1/2 - 2016/5/2
assets traded	Hang Seng china enterprises index futures
annualized return	107.00%
max drawdown	6.61%
Sharpe ratio	4.79

Optimal Trend Following (Math)

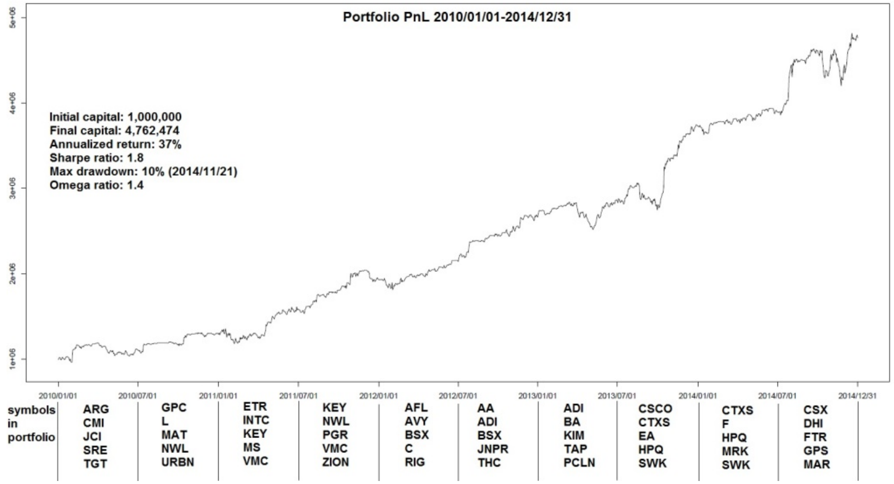
- Two state Markov model for a stock's prices: BULL and BEAR.
  - $dS_r = S_r\left[\mu_{\alpha_r} dr + \sigma dB_r\right],\ t \le r \le T < \infty$
  - The trading period is between time $[t, T]$.
  - $\alpha_r = \{1,2\}$ are the two Markov states that indicates the BULL and BEAR markets.
  - $\mu_1 > 0$
  - $\mu_2 < 0$
- $Q = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}$, the generator matrix for the Markov chain.
- When $i = 0$, expected return is
  - $\mathrm{E}_{0,t}(R_t) =$ $\mathrm{E}_t\left(e^{\rho(\tau_1 - t)} \prod_{n=1}^{\infty} \frac{S_{v_n}}{S_{\tau_n}} \left[\frac{1-K_s}{1+K_b}\right]^{I_{\{\tau_n < T\}}} e^{\rho(\tau_{n+1} - v_n)}\right)$
  - We are long between τ_n and v_n and the return is determined by the price change discounted by the commissions.
  - We are flat between v_n and τ_{n+1} and the money grows at the risk free rate.

- Value function:
  - $\mathrm{J}_0\,(S, \alpha, t, \Lambda_0) =$ $\mathrm{E}_t\left(\begin{array}{c}\rho(\tau_1 - t) + \\ \sum_{n=1}^{\infty}\left\{\log\frac{S_{v_n}}{S_{\tau_n}} + I_{\{\tau_n < T\}}\log\frac{1-K_s}{1+K_b} + \rho(\tau_{n+1} - v_n)\right\}\end{array}\right)$
  - $\mathrm{J}_1\,(S, \alpha, t, \Lambda_1) =$ $\mathrm{E}_t\left(\begin{array}{c}\left[\log\frac{S_{v_1}}{S} + \rho(\tau_2 - v_1) + \log(1 - K_s)\right] + \\ \sum_{n=2}^{\infty}\left\{\log\frac{S_{v_n}}{S_{\tau_n}} + I_{\{\tau_n < T\}}\log\frac{1-K_s}{1+K_b} + \rho(\tau_{n+1} - v_n)\right\}\end{array}\right)$
- Find an optimal trading sequence (the stopping times) so that the value functions are maximized.
  - $V_i(p, t) = \sup_{\Lambda_i} J_i(S, p, t, \Lambda_i)$
  - V_i: the maximum amount of expected returns
- $\begin{cases} V_0(p, t) = \sup_{\tau_1} E_t\{\rho(\tau_1 - t) - \log(1 + K_b) + V_1(p_{\tau_1}, \tau_1)\} \\ V_1(p, t) = \sup_{v_1} E_t\left\{\log\frac{S_{v_1}}{S_t} + \log(1 - K_s) + V_0(p_{v_1}, v_1)\right\} \end{cases}$
- Hamilton-Jacobi-Bellman Equations
  - $\begin{cases} \min\{-\mathcal{L}V_0 - \rho, V_0 - V_1 + \log(1 + K_b)\} = 0 \\ \min\{-\mathcal{L}V_1 - f(\rho), V_1 - V_0 - \log(1 - K_s)\} = 0 \end{cases}$
  - with terminal conditions: $\begin{cases} V_0(p, T) = 0 \\ V_1(p, T) = \log(1 - K_s) \end{cases}$
  - $\mathcal{L} = \partial_t + \frac{1}{2}\left(\frac{(\mu_1 - \mu_2)p(1-p)}{\sigma}\right)^2 \partial_{pp} + [-(\lambda_1 + \lambda_2)p + \lambda_2]\partial_p$

- Based on: M Dai, Q Zhang, QJ Zhu, "Trend following trading under a regime switching model," SIAM Journal on Financial Mathematics, 2010.

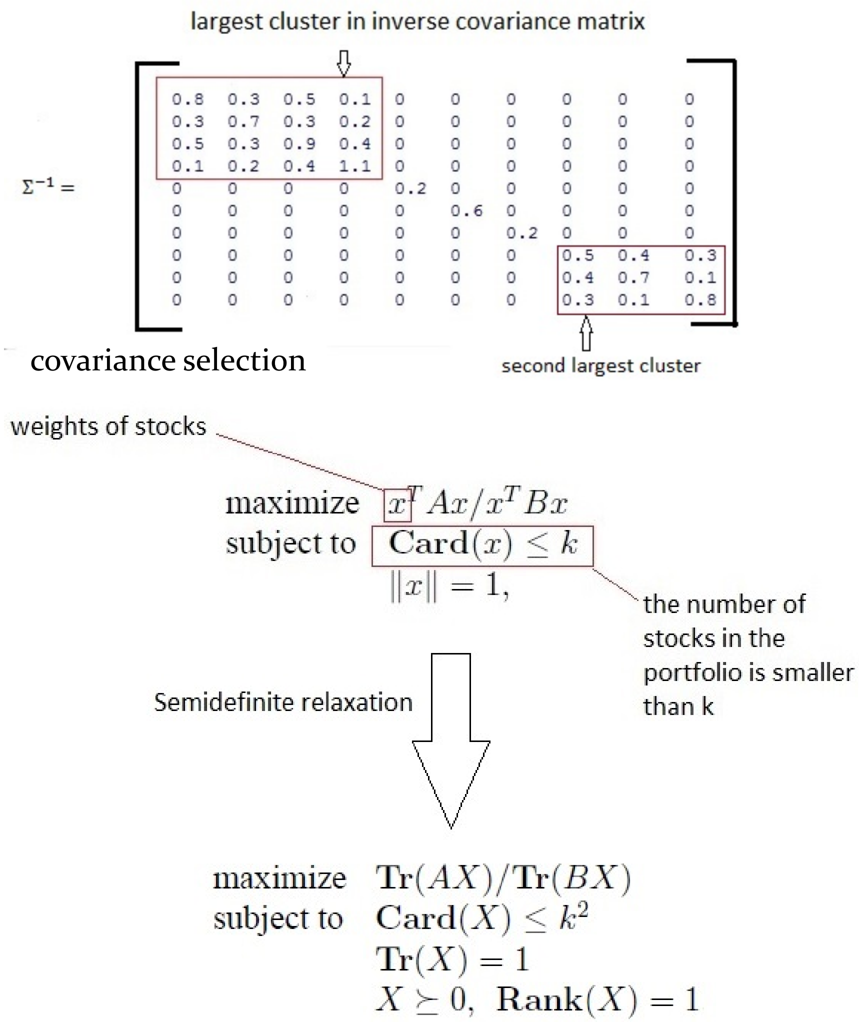
Optimal Mean Reversion (MR)

- Basket construction problem:
  - Select the right financial instruments.
  - Obtain the optimal weights for the selected financial instruments.
- Basket trading problem:
  - Given the portfolio can be modelled as a mean reverting OU process, dynamic spread trading is a stochastic optimal control problem.
  - Given a fixed amount of capital, dynamically allocate capital over a risky mean reverting portfolio and a risk-free asset over a finite time horizon to maximize a general constant relative risk aversion (CRRA) utility function of the terminal wealth .
  - Allocate capital amongst several mean reverting portfolios.



- Based on: Mudchanatongsuk, S., Primbs, J.A., Wong, " Optimal Pairs Trading: A Stochastic Control Approach," Dept. of Manage. Sci. & Eng., Stanford Univ., CA.

Optimal Mean Reversion (Math)

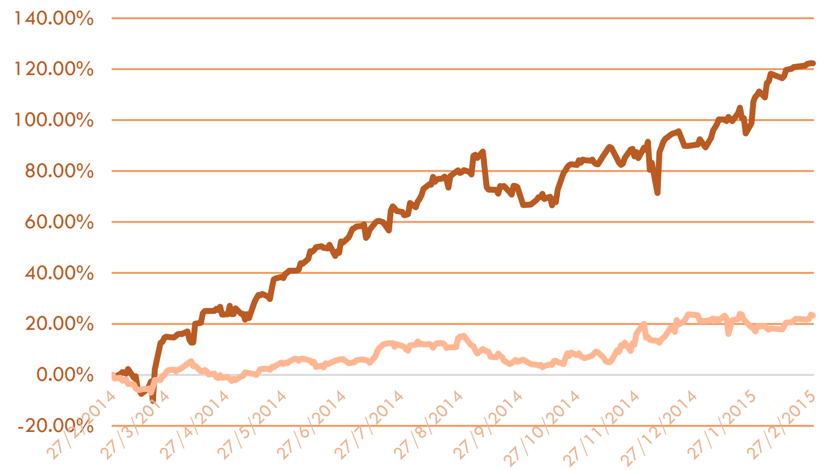


covariance selection

- Assume a risk free asset M_t, which satisfies
  - $dM_t = rM_t dt$
- Assume two assets, A_t and B_t.
- Assume B_t follows a geometric Brownian motion.
  - $dB_t = \mu B_t dt + \sigma B_t dz_t$
- x_t is the spread between the two assets.
  - $x_t = \log A_t - \log B_t$
- $\frac{dV_t}{V_t} = h_t \frac{dA_t}{A_t} + \widetilde{h_t} \frac{dB_t}{B_t} + \frac{dM_t}{M_t}$
- $= \left\{h_t \left[k(\theta - x_t) + \frac{1}{2}\eta^2 + \rho\eta\sigma\right] + r\right\} dt + h_t \eta d\omega_t$
- $\max_{h_t} E[V_T{}^\gamma]$, s.t.,
  - $V(0) = v_0$, $\mathrm{x}(0) = x_0$
  - $dx_t = k(\theta - x_t)dt + \eta d\omega_t$
  - $dV_t = h_t dx_t = h_t k(\theta - x_t)dt + h_t \eta d\omega_t$
- $h(t)^* = \frac{V_t}{(1-\gamma)}\left[-\frac{k}{\eta^2}(x_t - \theta) + 2\alpha(t)x_t + \beta(t)\right]$

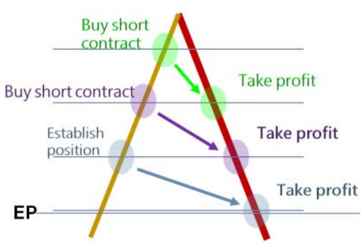
Intraday Volatility Trading (VOL)

- In mid or high frequency trading, or within a medium or short time interval, prices tend to oscillate.
- If there are enough oscillations before prices move in a direction, arbitrage exists.

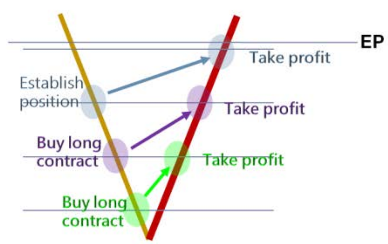


loss region

profit region



Above Equilibrium point



Below Equilibrium point

Live Result:

trading period	2014/2/27 - 2015/2/27
assets traded	Hang Seng china enterprises index futures
annualized return	122.32%
max drawdown	10.24%
Sharpe ratio	18.45

Intraday Volatility Trading (Math)

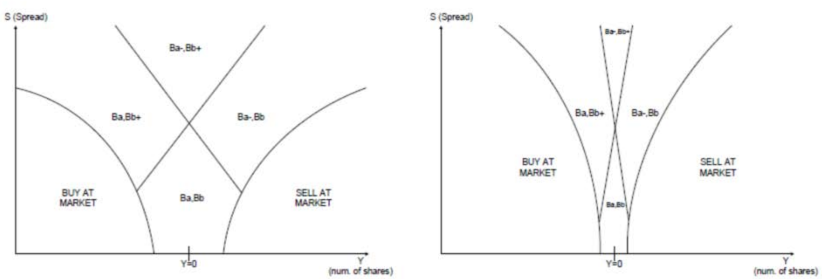
- For a continuous price process X_t, we define H-variation
  - $V_T(H,X) = \sup_T \sum_{l=1}^{L} |X(t_l) - X(t_{l-1})|$
- It can be shown that for any H, there exists a sequence $(\tau_n^*, \tau_n)_{n=0,1,\dots,N}$ such that $(\tau_n)_{n=0,1,\dots,N}$ are Markovian and τ_n^* are defined by X_t on intervals $[\tau_{n-1}, \tau_n]$. And they satisfy the equality:
  - $V_T(H,X) = \sum_{l=1}^{N} |X(\tau_n^*) - X(\tau_{n-1}^*)|$
- $N_T(H,X)$ is the number of KAGI-inversion in the T-interval.
- H-volatility:
  - $\eta_T(H,X) = \frac{V_T(H,X)}{N_T(H,X)}$
- For an no-arbitrage Wiener process, we have
  - $\lim_{T\to\infty} \eta_T(H,\sigma W) = KH = 2H$
- Define a trading strategy such that the position of X is:
  - $\tilde{\gamma}_t^K(H,X) = \sum_{n=1}^{N_T(H,X)} \operatorname{sign}\big(\chi(\tau_{n-1}) - \chi(\tau_{n-1}^*)\big)\, \chi_{[\tau_{n-1},\tau_n]}(t)$
- The trend following P&L is:
  - $\tilde{Y}_t^K(H,X) = \int_0^t \tilde{\gamma}_u^K(H,X)dX(u)$
  - $= (\eta_T(H,X) - 2H)N_T(H,X) + \varepsilon$
- The expected income per trade is:
  - $Y_t^K(H,X) = \int_0^t \gamma_u^K(H,X)dX(u)$
  - $y_t^K(H,X) = \frac{Y_t^K(H,X)}{N_t^K(H,X)}$
  - $\lim_{T\to\infty} \mathrm{E}\, y_t^K(H,X) = |K-2|H$

- Based on: SV Pastukhov, "On some probabilistic-statistical methods in technical analysis," Theory of Probability & Its Applications, SIAM, 2005.

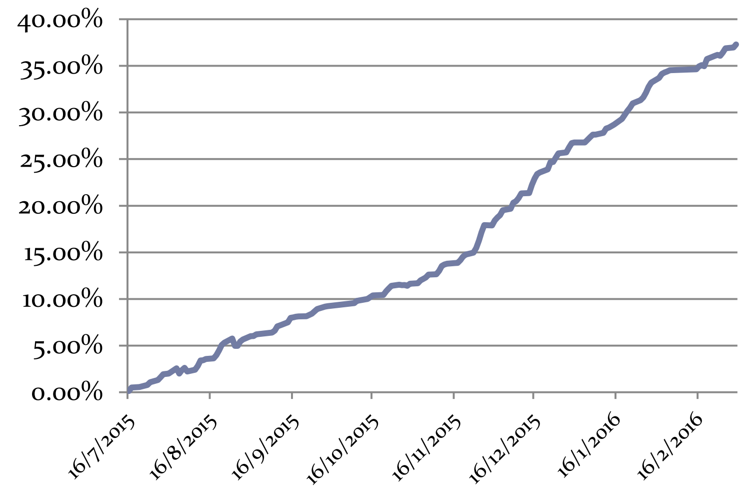
Optimal Market Making (MM)

- We optimally place limit and market orders depending on the current inventory and spread.

  the best market making strategy:



(a) near date 0 (b) near date T



Live Result:

trading period	2015/7/16 - 2016/3/1
assets traded	rebar + iron ore commodity futures
annualized return	65%
max drawdown	0.90%
Sharpe ratio	16.71

Optimal Market Making (Math)

- State variable:
  - (X, Y, P, S)
  - cash, inventory, mid price, spread
- Objective:
  - $\max_{\alpha} \mathrm{E}\left[U(X_T) - \gamma \int_0^T g(Y_t)dt\right]$
  - $Y_T = 0$, e.g., don't hold position overnight
  - U: utility function
  - X_T: terminal wealth
  - γ: penalty for holding inventory
- Liquidation function (how much we get by selling everything):
  - $L(x, y, p, s) = x - c(-y, p, s) = x + yp - |y|\frac{s}{2} - \varepsilon$
- Equivalent problem (get rid of $Y_T = 0$):
  - $\max_{\alpha} \mathrm{E}\left[U\big(L(X_T, Y_T, P_T, S_T)\big) - \gamma \int_0^T g(Y_t)dt\right]$
- Value function:
  - $v(t, z, s) = \sup_{\alpha} \mathop{\mathrm{E}}_{t,z,s}\left[U\big(L(Z_T, S_T)\big) - \gamma \int_t^T g(Y_u)du\right]$
    - $z = (x, y, p)$
- This is a mixed regular/impulse control problem in a regime switching jump-diffusion model.

- Quasi-Variational Inequality
  - $\min\left[-\frac{\partial v}{\partial t} - \sup \mathcal{L}^{q,l} v + \gamma g, v - \mathcal{M}v\right] = 0$
- Terminal condition:
  - $v(T, x, y, p, s) = U\big(L(x, y, p, s)\big)$
- For each state i, we have
  - $$\min\begin{bmatrix} -\frac{\partial v_i}{\partial t} - \mathcal{P}v_i - \sum_{j=1}^m r_{ij}(t)\left[v_j(t,x,y,p) - v_i(t,x,y,p)\right] \\ -\sup \lambda_i^b(q^b)\left[v_i\big(t, x - \pi_i^b(q^b,p)l^b, y + l^b, p\big) - v_i(t,x,y,p)\right] \\ -\sup \lambda_i^a(q^a)\left[v_i\big(t, x + \pi_i^a(q^a,p)l^a, y - l^a, p\big) - v_i(t,x,y,p)\right] \\ +\gamma g, \\ v_i(t,x,y,p) - \sup v_i\big(t, x - c_i(e,p), y + e, p\big) \end{bmatrix} = 0$$
  - $v_i(T, x, y, p) = U(L_i(x, y, p))$
- Assumptions:
  - $U(x) = x$; we care about only how much money made.
  - $(P_t)_t$ is a martingale; we know nothing about where the market will move.
- Solution:
  - $v_i(t, x, y, p) = x + yp + \phi_i(t, y)$
  - $\phi_i(t, y)$ is the solution to the system of integro-differential equations (IDE):
    - $$\min\begin{bmatrix} -\frac{\partial \phi_i}{\partial t} - \sum_{j=1}^m r_{ij}(t)\left[\phi_j(t,y) - \phi_i(t,y)\right] \\ -\sup \lambda_i^b(q^b)\left[\phi_i\big(t, y + l^b\big) - \phi_i(t,y) + \left(\frac{i\delta}{2} - \delta 1_{q^b = P_t^{b+}}\right) l^b\right] \\ -\sup \lambda_i^a(q^a)\left[\phi_i\big(t, y - l^a\big) - \phi_i(t,y) + \left(\frac{i\delta}{2} - \delta 1_{q^a = P_t^{a-}}\right) l^a\right] \\ +\gamma g(y), \\ \phi_i(t,y) - \sup\left[\phi_i(t, y + e) - \frac{i\delta}{2}|e| - \varepsilon\right] \end{bmatrix} = 0$$
    - $\phi_i(T, y) = -|y|\frac{i\delta}{2} - \varepsilon$

- Based on: F Guilbaud, H Pham, "Optimal high-frequency trading with limit and market orders," Quantitative Finance, 2013.

Conclusions

FinMath Infrastructure Support

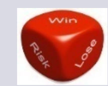
- All these mathematics and simulations are possible only with a finmath technology that serves as the modeling infrastructure.

Applications

- Trading Signals

- Asset Allocation

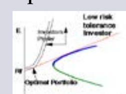
- Risk Management



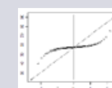
Financial Mathematics

- Optimal Trading Strategies

- Portfolio Optimization

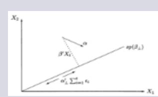


- Extreme Value Theory



Advanced Mathematics

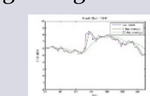
- Cointegration



- Time Series Analysis

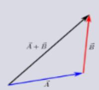
- Optimization (LP, QP, SQP, SDP, SOCP, IP, GA)

- Digital Signal Processing



Foundation Mathematics

- Linear algebra



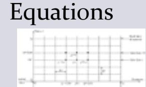
- Calculus

$$\int_a^b f(x) = F(b) - F(a)$$

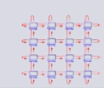
- Unconstrained optimization

- Statistics

- Differential Equations



- Parallelization



The Essential Skills

- Financial intuitions, market understanding, creativity.
- Mathematics.
- Computer programming.

An Emerging Field

- It is a financial industry where mathematics and computer science meet.
- It is an arms race to build
  - more reliable and faster execution platforms (computer science);
  - more comprehensive and accurate prediction models (mathematics).